ON THE SIMPLICIAL VOLUMES OF FIBER BUNDLES

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Abstract. We show that surface bundles over surfaces with base and fiber of genus at least 2 have non-vanishing simplicial volume.

The simplicial volume $||M||$, introduced by Gromov [3], is a homotopy invariant which measures the complexity of the fundamental class of an oriented manifold M. It is determined by the classifying map of the universal covering, and tends to be non-zero for large manifolds or fundamental groups, typically the negatively curved ones.

For products of compact oriented manifolds Gromov [3] proved that the simplicial volume is essentially multiplicative. More precisely, there are universal positive constants c_n depending only on $n = \dim(M_1 \times M_2)$ such that

$$||M_1 \times M_2|| \leq c_n ||M_1|| \cdot ||M_2|| \ , \tag{1}$$

and

$$||M_1 \times M_2|| \geq ||M_1|| \cdot ||M_2|| \ . \tag{2}$$

It is natural to wonder to what extent these inequalities hold for non-trivial fiber bundles instead of products. In this paper we address this question, initially posed to the second author by A. Landman.

The analog of the upper bound (1), which is elementary for products, fails for non-trivial bundles:

Example 1. There are hyperbolic 3-manifolds M which fiber over the circle. Like all hyperbolic manifolds, such M have non-zero simplicial volume, whereas $||S^1|| = 0$.

The lower bound (2), proved using bounded cohomology, actually holds for non-trivial bundles if the fiber has dimension 2:

Theorem 2. *Let X be the total space of a compact oriented fiber bundle with fiber a compact oriented surface F and compact oriented base*

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B (of arbitrary dimension). Then

$$||X|| \geq ||F|| \cdot ||B|| \ . \tag{3}$$

Proof. If F is a sphere or torus, $||F|| = 0$, so that there is nothing to prove. We may thus assume $g(F) \geq 2$.

Let $\pi \colon X \to B$ be the bundle projection, $e = e(T\pi) \in H^2(X)$ the Euler class of the tangent bundle along the fibers, and $\omega_B \in H^2(B)$ the fundamental class dual to the orientation class $[B] \in H_2(B)$. Let π_* denote integration along the fiber. Then

$$\langle e \cup \pi^* \omega_B, [X] \rangle = \langle \pi_*(e \cup \pi^* \omega_B), [B] \rangle = \langle e, [F] \rangle \cdot \langle \omega_B, [B] \rangle = \chi(F) \ .$$

Denoting by $|| \cdot ||_\infty$ the Gromov sup norm dual to the l^1 norm used in the definition of the simplicial volume, cf. [3], we deduce:

$$|\chi(F)| = ||e \cup \pi^* \omega_B||_\infty \cdot ||X|| \leq ||e||_\infty \cdot ||\omega_B||_\infty \cdot ||X|| \ ,$$

or

$$||X|| \geq \frac{1}{||e||_\infty} \cdot \frac{1}{||\omega_B||_\infty} \cdot |\chi(F)| \ .$$

As F is assumed to be a hyperbolic Riemann surface, we have $|\chi(F)| = \frac{1}{2}||F||$, see [3]. By definition, $\frac{1}{||\omega_B||_\infty} = ||B||$. Finally, the unit sphere bundle of $T\pi$ is a flat S^1-bundle [10], and so e is bounded. By the Milnor-Wood inequality, the exact bound is $||e||_\infty \leq \frac{1}{2}$ (cf. [2]), so that (3) follows. $\square$

Corollary 3. *Let X be the total space of an oriented surface bundle over a surface, with base F and fiber B both of genus ≥ 2. Then*

$$||X|| \geq 4\chi(X) > 0 \ . \tag{4}$$

Proof. This follows directly from (3), the equality $||\Sigma|| = -2\chi(\Sigma)$ for closed hyperbolic Riemann surfaces Σ, and the multiplicativity of the Euler characteristic in fiber bundles. $\square$

Remark 4. Combining (4) with the main result of [8], we obtain $||X|| \geq 8|\sigma(X)|$, where σ denotes the signature. However, using $3\sigma(X) = \langle e^2, [X] \rangle$ and $||e||_\infty \leq \frac{1}{2}$ as in the proof of Theorem 2, we obtain

$$||X|| \geq 12|\sigma(X)| \ .$$

Corollary 5. *Let X be the total space of a compact oriented fiber bundle with fiber F and base B. If dim$(X) \leq 4$, then*

$$||X|| \geq ||F|| \cdot ||B|| \ . \tag{5}$$

Proof. If $\dim(X) < 4$, then F or B must be a circle, so that the right-hand-side of (5) vanishes. Similarly, if $\dim(X) = 4$ and either F or B is a circle the right-hand-side of (5) vanishes and there is nothing to prove. Thus the only interesting case is that of a surface bundle, for which we appeal to Theorem 2. □

It is still an open problem whether there are surface bundles over surfaces admitting metrics of negative sectional curvature[1]. Nevertheless, Corollary 3 shows that as far as the simplicial volume is concerned, surface bundles look like negatively curved manifolds.

Example 6. Kapovich and Leeb [4] have given an example of a surface bundle over a surface with fiber and base both of genus ≥ 2 which does not admit any metric of non-positive curvature. By Corollary 3, the total space has non-vanishing simplicial volume.

As far as we know, this is the first example of an aspherical manifold with non-zero simplicial volume, but with no metric of non-positive curvature. If one does not insist on asphericity, examples can be constructed using connected sums, cf. [3].

Returning now to the upper bound (1) for the simplicial volume, we have seen already that there is no analog for fibered 3-manifolds. In the case of fibered 4-manifolds, we do not know if such an inequality holds in complete generality. However, if we make an additional geometric assumption, we have:

Proposition 7. *Let X be the total space of a compact oriented fiber bundle with fiber F and base B. If $\dim(X) = 4$, and X admits an Einstein metric, then*

$$||X|| \leq c \cdot ||F|| \cdot ||B|| \ , \tag{6}$$

for some universal positive constant c.

Proof. As X is an Einstein manifold, we have Berger's inequality $\chi(X) \geq 0$, with equality only if X is flat. Flat manifolds are finitely covered by the torus, and so their simplicial volumes vanish and there is nothing to prove in that case. Thus we can assume $\chi(X) > 0$. This means that neither F nor B can be a circle or a 2-torus. Moreover, if F is a 2-sphere, then so is B, and vice versa. In this case both sides of (6) vanish.

Thus, we only have to consider the case when both F and B are hyperbolic Riemann surfaces. Then $||F|| \cdot ||B|| = 4\chi(F) \cdot \chi(B) = 4\chi(X)$, so that we only have to show that the simplicial volume of X is

[1] Compare Problem 2.12 (A) in Kirby's list [5].

bounded above by a universal multiple of its Euler characteristic. But this follows from the Gromov-Hitchin-Thorpe inequality

$$\chi(X) \geq \frac{3}{2}|\sigma(X)| + \frac{1}{2592\pi^2}||X||$$

for Einstein 4-manifolds proved in [3, 7]. □

Remark 8. The Proposition applies in particular to the surface bundles over surfaces (with $g(F), g(B) \geq 2$) which admit complex structures, such as the ones constructed by Atiyah [1] and by Kodaira [6], as they carry Kähler-Einstein metrics due to the results of Aubin and Yau on the Calabi conjecture.

The fact that results on surface bundles tend to be stronger when the total space is complex or admits an Einstein metric is already familiar from [8, 9].

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